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Response of a relativistic quantum magnetized electron gas

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Abstract

The response 4-tensor is derived for a spin-independent, relativistic magnetized quantum electron gas. The sum over spins is carried out both directly and using a procedure due to Ritus. The 4-tensor components are written in terms of a sum over the two solutions of the resonance condition for the particle 4-momentum. It is shown that the dispersive properties may be described in terms of a single plasma dispersion function, for arbitrary occupation numbers for electrons and positrons in each Landau level. The plasma dispersion function is evaluated explicitly in the completely degenerate and nondegenerate thermal limits. The perpendicular wave number appears in the arguments of J-functions, which are proportional to generalized Laguerre polynomials, but not in the plasma dispersion function. The result generalizes a known form for the response tensor for parallel propagation (in the completely degenerate case), when the J-functions are either zero or unity, to arbitrary angles of propagation.

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1. Introduction

From a formal point of view, the most general description of the response of a (collisionless) electron gas is that based on relativistic quantum theory. General forms for the response tensor derived using quantum electrodynamics (QED) have long been available in both the unmagnetized case [1–6], cf also [7], and the magnetized case [8–16], cf also [4, 17, 18]. A complicating feature of the magnetized case is the spin of the electron. In the unmagnetized case, the use of the Feynman propagator allows one to write down the response 4-tensor, denoted by $\Pi^{\mu\nu}(k)$ with k^{μ} being the wave 4-vector, for any unpolarized electron gas without considering the spin explicitly. Specific results have long been available for the completely degenerate limit [1, 2, 4, 5], and also for the nondegenerate limit [3], corresponding to a Jüttner distribution. In contrast, the treatment of the spin is a seemingly unavoidable major complication in the magnetized case. A specific calculation involves constructing the Dirac wavefunction for a magnetized electron, evaluating the response tensor (by one of several possible methods) and then performing the sums over spins term by term. Explicit results are available only for special cases, notably parallel [4, 10, 11, 13, 16, 18] and longitudinal [14] propagation in a completely degenerate electron gas. There are applications for which it is desirable to have a more general expression for the response tensor, notably in connection with neutrino emission from the cores of compact magnetized stars [19], where only parallel propagating waves in the completely degenerate limit have been considered, and to pulsar and magnetar magnetospheres [20], where the use of classically derived response tensors precludes discussion of intrinsically relativistic quantum effects.

Our objective in this paper is to derive the relativistic quantum expression of the response tensor for an arbitrary magnetized electron gas, subject only to the restriction that the electron occupation number at each Landau level, n, is independent of spin. We start from a known general expression for the response tensor that involves explicit sums over electron and positron states described by quantum numbers $\epsilon = \pm 1$, for electrons and positrons, respectively, $s = \pm$ describing the spin, and n; there is also a sum over an intermediate state described by ϵ', s', n' and an integral over the parallel momentum p_z. The first step in the calculation is to perform the sums over s, s', to derive each component of the tensor as a dispersion integral over p_z that involves the occupation number $n_n^\epsilon(p_z)$. We perform the sums over s, s' in two different ways. One way is to use specific spin-dependent expressions, written down in appendix A, and perform the sums over s, s' explicitly. The other way is based on the Ritus method [21], outlined in appendix B, in which the sum over spin states is effectively replaced by traces over Dirac matrices, as in the unmagnetized case for unpolarized electrons. A complicating feature arises from the eigenstates for a specific n involving simple-harmonic wavefunctions with quantum number $l = n - (s+1)/2$, requiring that one separates the state with $n = l$ from that with $n - 1 = l$. As a result, the traces include projection operators onto two subspaces; the conventional rules for evaluating traces are generalized to incorporate these projections in appendix B. Both methods are cumbersome; the Ritus method has the advantage that it leads directly to a 4-tensor form for the response tensor.

A second step in the calculation involves rationalizing the resonant denominator and summing over ϵ'. The resonant denominator is $\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'}$, and rationalization involves removing the square roots in the energies $\varepsilon_n = \left(m^2 + p_z^2 + 2neB\right)^{1/2}, \varepsilon'_{n'} = \left(m^2 + p_z'^2 + 2n'eB\right)^{1/2}$, where we use natural units ($\hbar = c = 1$) with B being the magnetic field. The rationalized denominator is a quadratic function of p_z, $\propto (p_z - p_{z+})(p_z - p_{z-})$ say, allowing each dispersion integral to be written as a sum of integrals with denominators $p_z - p_{z\pm}$. The response 4-tensor is reduced to a sum over $\pm$ of tensor components involving the particle 4-momentum, $p_\pm^\mu$, evaluated at the resonance; it also involves generalized Laguerre polynomials, written as functions $J_{n-n'}^n(x)$, with $x = k_\perp^2/2eB$, and the dispersion integrals. The general result, although cumbersome, can be written in a relatively concise form. Major simplification to the J-functions occurs for $x \ll 1$, which is the case for small angles of propagation or for sufficiently strong magnetic fields.

A third step is to show how all dispersion integrals may be expressed in terms of a single dispersion function, which depends only on $n_n^\epsilon(p_z)$ and is independent of the perpendicular wave number, $k_\perp$. This leads to a new unifying result: one may write down the general form of the response tensor in terms of this dispersion function and evaluate the dispersion function separately. We note that the existing results for parallel propagation in the completely degenerate limit already contain this dispersion function (a logarithmic function in this case) for a completely degenerate distribution. Our generalization shows that the same dispersion function applies without modification to the oblique case, $k_\perp \neq 0$, so that the dispersive properties discussed for parallel propagation [18] also apply for oblique propagation. For a (magnetized) Jüttner distribution, the dispersion function can be expressed in terms of a plasma dispersion function introduced for a Jüttner distribution in the nonquantum, unmagnetized case [7]. We also comment on the comparison of our results with some known results, including the ultrarelativistic and nonrelativistic quantum limits.

In section 2, general results for the response tensor are written down for a spin-independent electron gas. In section 3, the sum over intermediate states is reduced to a single sum over the Landau quantum number of the virtual electron or positron. Specific plasma dispersion functions are defined in section 4, where it is shown that a single plasma dispersion function suffices for any given electron distribution. In sections 5 and 6, we compare our results with some known results in the completely degenerate limit and the nondegenerate limit, respectively. The results are discussed in section 7 and the conclusions are summarized in section 8.

2. General forms for the response tensor

In this section, a general form for the response tensor is derived for a spin-independent electron gas.

2.1. Derivation using the vertex formalism

A conventional momentum-space form of QED is not possible in the presence of a magnetic field. An alternative is provided by a vertex formalism [15], in which the spatial dependence of the wavefunctions associated with each vertex is represented in terms of a Fourier-transformed vertex function. The vertex function depends on the choice of a spin operator, and the response tensor for a spin-independent electron gas is found by choosing a spin operator, constructing the vertex functions and summing the known form for the response tensor in the vertex formalism over the spins. This gives

$$\Pi^{\mu\nu}(k) = -\frac{e^3 B}{2\pi}\sum_{\epsilon,n,\epsilon',n'}\int\frac{\mathrm{d}p_z}{2\pi}\int\frac{\mathrm{d}p'_z}{2\pi}2\pi\,\delta(\epsilon' p'_z - \epsilon p_z + k_z)$$
$$\times\,\frac{\frac{1}{2}(\epsilon'-\epsilon)+\epsilon n_n^{\epsilon}(p_z)-\epsilon' n_{n'}^{\epsilon'}(p'_z)}{\omega-\epsilon\varepsilon_n+\epsilon'\varepsilon'_{n'}+\mathrm{i}0}\,\frac{[C_{n'n}(\epsilon' p'_\parallel,\epsilon p_\parallel)]^{\mu\nu}}{2\epsilon'\epsilon\varepsilon'_{n'}\varepsilon_n}, \tag{1}$$

where ε_n denotes $\varepsilon_n(p_z) = \left(m^2+p_z^2+p_n^2\right)^{1/2}$, $p_n = (2neB)^{1/2}$ and $\varepsilon'_{n'}$ denotes $\varepsilon_{n'}(p'_z)$, and with $p_\parallel^\mu = (\varepsilon_n, 0, 0, p_z)$, $p_\parallel'^\mu = (\varepsilon'_{n'}, 0, 0, p'_z)$. The resonance condition is imposed by (the Landau prescription) giving the frequency an infinitesimal positive imaginary part, $\omega \to \omega + \mathrm{i}0$, in the denominator. The electron gas is described by the occupation numbers, $n_n^\epsilon(p_z)$, for electrons and positrons. In the absence of an electron gas, the response tensor (1) gives the unregularized response of the magnetized vacuum: the vacuum response tensor is derived from it by an appropriate regularization procedure.

The tensor in the integrand in (1) is defined by the sum

$$\frac{[C_{n'n}(\epsilon' p'_\parallel,\epsilon p_\parallel)]^{\mu\nu}}{2\epsilon'\epsilon\varepsilon'_{n'}\varepsilon_n} = \sum_{s,s'}\left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^\mu\left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^{*\nu}, \tag{2}$$

where $\left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^\mu$ is the vertex function, with the quantum numbers q denoting p_z, n, s and q' denoting p'_z, n', s'. An explicit form for the vertex function is written down in appendix A for the specific choice of the (z-component of the) magnetic moment as the spin operator. The coordinate system is chosen such that the magnetic field is along the z-axis, with $\mathbf{k} = (k_\perp\cos\psi, k_\perp\sin\psi, k_z)$.

For some purposes it is convenient to choose a set of basis 4-vectors that allows the tensor (2), and hence the response tensor itself, to be written in terms of invariant components with respect to these basis vectors. One such choice is made as follows. Using the Maxwell tensor for the background magnetic field, $F^{\mu\nu} = Bf^{\mu\nu}$, one may separate the metric tensor, $g^{\mu\nu} = g_\parallel^{\mu\nu} + g_\perp^{\mu\nu}$, into parts that span the 2D $\parallel$-subspace defined by the time-**B** axes and the orthogonal 2D $\perp$-subspace, with $g_\perp^{\mu\nu} = -f^\mu{}_\alpha f^{\alpha\nu}$. On combining $f^{\mu\nu}$ and its dual, $\phi^{\mu\nu}$, with the wave 4-vector, k^μ, one may construct a set of four orthogonal 4-vectors that span both subspaces. Such a set is

$$k_\parallel^\mu = g_\parallel^{\mu\nu} k_\nu = (\omega, 0, 0, k_z), \qquad k_\perp^\mu = g_\perp^{\mu\nu} k_\nu = (0, k_\perp \cos\psi, k_\perp \sin\psi, 0),$$
$$k_G^\mu = -f^{\mu\nu} k_\nu = (0, -k_\perp \sin\psi, k_\perp \cos\psi, 0), \qquad k_D^\mu = \phi^{\mu\nu} k_\nu = (k_z, 0, 0, \omega). \tag{3}$$

The Onsager relations imply that there are only six independent components of the response 3-tensor (three diagonal and three off-diagonal). Similarly, for the response 4-tensor there are only six independent components: in this case, the Onsager relations imply that there are only ten independent components (four diagonal and six off-diagonal), and the charge-continuity or gauge-invariance relations give four additional constraints, implying that the components along $k^\mu = k_\parallel^\mu + k_\perp^\mu$, or k^ν are identically zero.

2.2. Derivation using the Ritus method

An alternative way of deriving the response tensor is by using the Ritus method, in which the sum over spins is performed implicitly, and the explicit evaluation involves traces over Dirac matrices. Performing the relevant traces is more cumbersome than in the unmagnetized case; relevant sums are summarized in appendix B. The results of both the Ritus method and the direct sum (2) can be written in the form

$$\begin{aligned}
\left[C_{n'n}^{\epsilon'\epsilon}(P'_\parallel, P_\parallel)\right]^{\mu\nu} = {} & \left\{P_\parallel'^\mu P_\parallel^\nu + P_\parallel^\mu P_\parallel'^\nu - \left[(P'P)_\parallel - m^2\right] g_\parallel^{\mu\nu}\right\}\left[\left(J_{n'-n}^{n-1}\right)^2 + \left(J_{n'-n}^{n}\right)^2\right] \\
& - \left[(P'P)_\parallel - m^2\right]\left\{g_\perp^{\mu\nu}\left[\left(J_{n'-n+1}^{n-1}\right)^2 + \left(J_{n'-n-1}^{n}\right)^2\right] - i f^{\mu\nu}\left[\left(J_{n'-n+1}^{n-1}\right)^2\right.\right. \\
& \left.\left. - \left(J_{n'-n-1}^{n}\right)^2\right]\right\} + p_{n'} p_n \left\{g_\parallel^{\mu\nu} 2 J_{n'-n}^{n-1} J_{n'-n}^{n} + \left(e_+^\mu e_+^\nu + e_-^\mu e_-^\nu\right) J_{n'-n+1}^{n-1} J_{n'-n-1}^{n}\right\} \\
& - p_n P_\parallel'^\mu \left[J_{n'-n}^{n-1} J_{n'-n-1}^{n} e_+^\nu + J_{n'-n}^{n} J_{n'-n+1}^{n-1} e_-^\nu\right] \\
& - p_n P_\parallel'^\nu \left[J_{n'-n}^{n-1} J_{n'-n-1}^{n} e_-^\mu + J_{n'-n}^{n} J_{n'-n+1}^{n-1} e_+^\mu\right] \\
& - p_{n'} P_\parallel^\mu \left[J_{n'-n}^{n-1} J_{n'-n+1}^{n-1} e_-^\nu + J_{n'-n}^{n} J_{n'-n-1}^{n} e_+^\nu\right] \\
& - p_{n'} P_\parallel^\nu \left[J_{n'-n}^{n-1} J_{n'-n+1}^{n-1} e_+^\mu + J_{n'-n}^{n} J_{n'-n-1}^{n} e_-^\mu\right].
\end{aligned} \tag{4}$$

The 4-vectors introduced in (4) are defined by $P_\parallel^\mu = (\epsilon\varepsilon_n, 0, 0, \epsilon p_z)$, $P_\parallel'^\mu = (\epsilon'\varepsilon'_{n'}, 0, 0, \epsilon' p'_z)$, $e_\pm^\mu = (k_\perp^\mu \pm \mathrm{i} k_G^\mu)/k_\perp$. The J-functions are defined by

$$J_\nu^n(x) = (-)^\nu J_{-\nu}^{n+\nu}(x) = \left(\frac{n!}{(n+\nu)!}\right)^{1/2} \mathrm{e}^{-x/2} x^{\nu/2} L_n^\nu(x), \tag{5}$$

where $L_n^\nu(x)$ is the generalized Laguerre polynomial (A.2). The argument $x = k_\perp^2/2eB$ of the J-functions is omitted in (4).

The tensor (4) does not satisfy the charge-continuity and gauge-invariance relations; rather it satisfies the identities

$$\begin{aligned}
k_\mu\left[C_{n'n}^{\epsilon'\epsilon}(P'_\parallel, P_\parallel)\right]^{\mu\nu} &= (\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'})\left[C_{n'n}^{\epsilon'\epsilon}(P'_\parallel, P_\parallel)\right]^{0\nu}, \\
k_\nu\left[C_{n'n}^{\epsilon'\epsilon}(P'_\parallel, P_\parallel)\right]^{\mu\nu} &= (\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'})\left[C_{n'n}^{\epsilon'\epsilon}(P'_\parallel, P_\parallel)\right]^{\mu 0}.
\end{aligned} \tag{6}$$

Relations (6) suffice to ensure that the response tensor itself satisfies the charge-continuity and gauge-invariance relations: the factors in parentheses on the right-hand sides of (6) cancel with the resonant denominators in (1), and the resulting expressions integrate to zero. To establish that the form (4) satisfies (6), one needs the relations

$$\begin{aligned} p_{n'} J^{n}_{n'-n}(x) &= p_n J^{n-1}_{n'-n}(x) + k_\perp J^{n}_{n'-n-1}(x), \\ p_{n'} J^{n-1}_{n'-n}(x) &= p_n J^{n}_{n'-n}(x) + k_\perp J^{n-1}_{n'-n+1}(x). \end{aligned} \tag{7}$$

3. Sum over intermediate electron and positron states

In this section, the sum over the electron and positron contributions to the virtual intermediate state is performed explicitly, and the result is written in terms of resonant values of p_z.

3.1. Resonant values of p_z

The denominator, $\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'}$, in (1) can be rationalized to remove the square roots by multiplying numerator and denominator by $\omega - \epsilon\varepsilon_n - \epsilon'\varepsilon'_{n'}$, $\omega + \epsilon\varepsilon_n - \epsilon'\varepsilon'_{n'}$, $\omega + \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'}$. The denominator becomes

$$\begin{aligned} D(\omega, \varepsilon_n, \varepsilon'_{n'}) &= (\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'})(\omega + \epsilon\varepsilon_n - \epsilon'\varepsilon'_{n'})(\omega - \epsilon\varepsilon_n - \epsilon'\varepsilon'_{n'})(\omega + \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'}) \\ &= -4\big(\omega^2 - k_z^2\big)(\epsilon p_z - p_{z+})(\epsilon p_z - p_{z-}) \\ &= -4\big(\omega^2 - k_z^2\big)(\epsilon' p'_z - p'_{z+})(\epsilon' p'_z - p'_{z-}). \end{aligned} \tag{8}$$

The two alternative forms for (8) follow by noting that it can be written as a quadratic form in either p_z or p'_z and solved for the resonant values [12].

Explicit forms for the resonant momenta are

$$p_{z\pm} = k_z f_{nn'} \pm \omega g_{nn'}, \qquad p'_{z\pm} = k_z(f_{nn'} - 1) \pm \omega g_{nn'}, \tag{9}$$

which depend on n, n' through

$$f_{nn'} = \frac{\big(\varepsilon_n^0\big)^2 - \big(\varepsilon_{n'}^0\big)^2 + \omega^2 - k_z^2}{2\big(\omega^2 - k_z^2\big)}, \tag{10}$$

$$g^2_{nn'} = \frac{\big[\omega^2 - k_z^2 - \big(\varepsilon_n^0 - \varepsilon_{n'}^0\big)^2\big]\big[\omega^2 - k_z^2 - \big(\varepsilon_n^0 + \varepsilon_{n'}^0\big)^2\big]}{4\big(\omega^2 - k_z^2\big)^2}, \tag{11}$$

with $\varepsilon_n^0 = \big(m^2 + p_n^2\big)^{1/2}$, and where the identity $(p^2)_\parallel = \big(\varepsilon_n^0\big)^2$ implies $\big(\omega^2 - k_z^2\big)\big(f^2_{nn'} - g^2_{nn'}\big) = \big(\varepsilon_n^0\big)^2$. The resonant energies are

$$\varepsilon_\pm = \varepsilon_n(p_{z\pm}) = \omega f_{nn'} \pm k_z g_{nn'}, \qquad \varepsilon'_\pm = \varepsilon_{n'}(p'_{z\pm}) = \omega(f_{nn'} - 1) \pm k_z g_{nn'}. \tag{12}$$

3.2. Sums over ϵ', ϵ

The sum over ϵ' in the p_z-integral in (1), and the sum over ϵ in the p'_z-integral, can be performed to give

$$\begin{aligned} \sum_{\epsilon'} \frac{\epsilon'[C_{n'n}(P'_\parallel, P_\parallel)]^{\mu\nu}}{2\varepsilon'_{n'}\varepsilon_n(\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'})} &= \frac{[C_{n'n}(P_\parallel - k_\parallel, P_\parallel)]^{\mu\nu}}{2\varepsilon_n\big[\epsilon(\varepsilon_n\omega - p_z k_z) - \big(\omega^2 - k_z^2\big) f_{nn'}\big]}, \\ \sum_{\epsilon} \frac{\epsilon[C_{n'n}(P'_\parallel, P_\parallel)]^{\mu\nu}}{2\varepsilon'_{n'}\varepsilon_n(\omega - \epsilon\varepsilon_n + \epsilon'\varepsilon'_{n'})} &= \frac{[C_{n'n}(P'_\parallel, P'_\parallel + k_\parallel)]^{\mu\nu}}{2\varepsilon'_{n'}\big[\epsilon'(\varepsilon'_{n'}\omega - p'_z k_z) + \big(\omega^2 - k_z^2\big) f_{n'n}\big]}, \end{aligned} \tag{13}$$

respectively. Expression (1) for the response tensor becomes

$$\Pi^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{n,n'}\bigg[\sum_{\epsilon}\int\frac{\mathrm{d}p_z}{2\pi}n_n^{\epsilon}(p_z)\frac{[C_{n'n}(P_\parallel - k_\parallel, P_\parallel)]^{\mu\nu}}{2\varepsilon_n\big[\epsilon(\varepsilon_n\omega - p_zk_z) - \big(\omega^2 - k_z^2\big)f_{nn'}\big]}$$
$$-\sum_{\epsilon'}\int\frac{\mathrm{d}p'_z}{2\pi}n_{n'}^{\epsilon'}(p'_z)\frac{[C_{n'n}(P'_\parallel, P'_\parallel + k_\parallel)]^{\mu\nu}}{2\varepsilon'_{n'}\big[\epsilon'(\varepsilon'_{n'}\omega - p'_zk_z) + \big(\omega^2 - k_z^2\big)f_{n'n}\big]}\bigg]. \quad (14)$$

To combine the two integrals in (14) into a single integral with the same resonant denominator, one rewrites the p'_z-integral by making the changes $p'_z \leftrightarrow p_z$, $n' \leftrightarrow n$, $\epsilon' \leftrightarrow -\epsilon$. The resulting expression is

$$\Pi^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{\epsilon,n,n'}\int\frac{\mathrm{d}p_z}{2\pi}\frac{\epsilon(\varepsilon_n\omega + p_zk_z) + \big(\omega^2 - k_z^2\big)f_{nn'}}{2\varepsilon_n\big(\omega^2 - k_z^2\big)(\epsilon p_z - p_{z+})(\epsilon p_z - p_{z-})}$$
$$\times\big[n_n^{\epsilon}(p_z)[C_{n'n}(P_\parallel - k_\parallel, P_\parallel)]^{\mu\nu} + n_n^{-\epsilon}(p_z)[C_{nn'}(-P_\parallel, -P_\parallel + k_\parallel)]^{\mu\nu}\big]. \quad (15)$$

3.3. Charge-symmetric and anti-symmetric parts

The occupation numbers in (15) may be written in terms of the sum and difference of the electron and positron contributions: $n_n^{\epsilon}(p_z) = \frac{1}{2}\big[\bar{n}_n(p_z) + \epsilon n_n^d(p_z)\big]$ with

$$\bar{n}_n(p_z) = n_n^+(p_z) + n_n^-(p_z), \qquad n_n^d(p_z) = n_n^+(p_z) - n_n^-(p_z). \quad (16)$$

The integrand then separates naturally into nongyrotropic and gyrotropic parts, which are symmetric and anti-symmetric, respectively, under the interchange of electrons and positrons:

$$\Pi^{\mu\nu}(k) = -\frac{e^3B}{2\pi}\sum_{\epsilon,n,n'}\int\frac{\mathrm{d}p_z}{2\pi}\frac{\epsilon(\varepsilon_n\omega + p_zk_z) + \big(\omega^2 - k_z^2\big)f_{nn'}}{2\varepsilon_n\big(\omega^2 - k_z^2\big)(\epsilon p_z - p_{z+})(\epsilon p_z - p_{z-})}$$
$$\times\big[\bar{n}_n(p_z)[N_{n'n}(\epsilon p_\parallel, k)]^{\mu\nu} + \epsilon n_n^d(p_z)[G_{n'n}(\epsilon p_\parallel, k)]^{\mu\nu}\big], \quad (17)$$

where the argument $\epsilon p_\parallel$ of N and G denotes the 4-vector $P_\parallel^\mu = \epsilon p_\parallel^\mu$. The nongyrotropic part is given by

$$[N_{n'n}(\epsilon p_\parallel, k)]^{\mu\nu} = [\epsilon(pk)_\parallel - 2neB]\big\{g_\parallel^{\mu\nu}\big[\big(J_{n'-n}^{n-1}\big)^2 + \big(J_{n'-n}^{n}\big)^2\big] + g_\perp^{\mu\nu}\big[\big(J_{n'-n+1}^{n-1}\big)^2$$
$$+\big(J_{n'-n-1}^{n}\big)^2\big]\big\} + [2p_\parallel^\mu p_\parallel^\nu - (\epsilon p_\parallel^\mu k_\parallel^\nu + \epsilon p_\parallel^\nu k_\parallel^\mu)]\big[\big(J_{n'-n}^{n-1}\big)^2 + \big(J_{n'-n}^{n}\big)^2\big]$$
$$+2p_{n'}p_n\big\{g_\parallel^{\mu\nu}J_{n'-n}^{n-1}J_{n'-n}^{n} + \big[e_1^\mu e_1^\nu - e_2^\mu e_2^\nu\big]J_{n'-n+1}^{n-1}J_{n'-n-1}^{n}\big\}$$
$$+p_n\big(k_\parallel^\mu e_1^\nu + k_\parallel^\nu e_1^\mu\big)\big[J_{n'-n}^{n-1}J_{n'-n-1}^{n} + J_{n'-n}^{n}J_{n'-n+1}^{n-1}\big]$$
$$-p_n\big[\epsilon p_\parallel^\mu e_1^\nu + \epsilon p_\parallel^\nu e_1^\mu\big]\big[J_{n'-n}^{n-1}J_{n'-n-1}^{n} + J_{n'-n}^{n}J_{n'-n+1}^{n-1}\big]$$
$$-p_{n'}\big[\epsilon p_\parallel^\mu e_1^\nu + \epsilon p_\parallel^\nu e_1^\mu\big]\big[J_{n'-n}^{n-1}J_{n'-n+1}^{n-1} + J_{n'-n}^{n}J_{n'-n-1}^{n}\big], \quad (18)$$

and the gyrotropic part is given by

$$[G_{n'n}(\epsilon p_\parallel, k)]^{\mu\nu} = \mathrm{i}\big\{f^{\mu\nu}[2neB - \epsilon(pk)_\parallel]\big[\big(J_{n'-n+1}^{n-1}\big)^2 - \big(J_{n'-n-1}^{n}\big)^2\big]$$
$$+p_n\big(k_\parallel^\mu e_2^\nu - k_\parallel^\nu e_2^\mu\big)\big[J_{n'-n}^{n-1}J_{n'-n-1}^{n} - J_{n'-n}^{n}J_{n'-n+1}^{n-1}\big]$$
$$-p_n\big[\epsilon p_\parallel^\mu e_2^\nu - \epsilon p_\parallel^\nu e_2^\mu\big]\big[J_{n'-n}^{n-1}J_{n'-n-1}^{n} - J_{n'-n}^{n}J_{n'-n+1}^{n-1}\big]$$
$$+p_{n'}\big[\epsilon p_\parallel^\mu e_2^\nu - \epsilon p_\parallel^\nu e_2^\mu\big]\big[J_{n'-n}^{n-1}J_{n'-n+1}^{n-1} - J_{n'-n}^{n}J_{n'-n-1}^{n}\big]\big\}, \quad (19)$$

with $e_1^\mu = k_\perp^\mu/k_\perp$, $e_2^\mu = -f^{\mu\nu}k_{\perp\nu}/k_\perp = k_G^\mu/k_\perp$, $k_\parallel^\mu = (\omega, 0, 0, k_z)$ and where the argument, $x = k_\perp^2/2eB$, of the J-function is omitted.

4. Dispersion integrals

The integral over p_z in expression (17) may be interpreted as a dispersion integral that depends on the particular form for the occupation numbers $n_n^\epsilon(p_z)$. We show that the dispersion integrals can be expressed in terms of a single dispersion integral, which becomes a plasma dispersion function for any given $n_n^\epsilon(p_z)$.

4.1. Reduction of dispersion integrals

The integral over p_z in (17) may first be expressed in terms of two dispersion integrals. After writing the denominator in (17) using

$$\frac{1}{(\epsilon p_z - p_{z+})(\epsilon p_z - p_{z-})} = \frac{1}{2\omega g_{nn'}} \sum_{\pm} \frac{\pm 1}{\epsilon p_z - p_{z\pm}}, \tag{20}$$

the p_z-integral may be evaluated in terms of the two integrals

$$I_{1\pm}(\omega, k_z) = \int \mathrm{d}p_z \frac{n_n(p_z)}{\epsilon p_z - p_{z\pm}}, \qquad I_{2\pm}(\omega, k_z) = \varepsilon_n^0 \int \frac{\mathrm{d}p_z}{\varepsilon_n} \frac{n_n(p_z)}{\epsilon p_z - p_{z\pm}}. \tag{21}$$

The functions defined by (21) are written as $I_{i\pm}(\omega, k_z) \to \bar{I}_{i\pm}(\omega, k_z), I_{i\pm}^d(\omega, k_z)$ for $n_n(p_z) \to \bar{n}_n(p_z), n_n^d(p_z)$, respectively, with $i = 1, 2$. On inserting (18), (19) into (17), the integrals required are

$$\begin{pmatrix} H(\omega, k_z) \\ H^\mu(\omega, k_z) \\ H^{\mu\nu}(\omega, k_z) \end{pmatrix} = \int \mathrm{d}p_z \frac{n_n(p_z)}{\varepsilon_n} \frac{\epsilon(\varepsilon_n\omega + p_z k_z) + \left(\omega^2 - k_z^2\right) f_{nn'}}{(\epsilon p_z - p_{z+})(\epsilon p_z - p_{z-})} \begin{pmatrix} 1 \\ \epsilon p_\parallel^\mu \\ p_\parallel^\mu p_\parallel^\nu \end{pmatrix}. \tag{22}$$

One finds

$$\begin{pmatrix} H(\omega, k_z) \\ H^\mu(\omega, k_z) \\ H^{\mu\nu}(\omega, k_z) \end{pmatrix} = \sum_{\pm} \pm K_\pm(\omega, k_z) \begin{pmatrix} 1 \\ p_\pm^\mu \\ p_\pm^\mu p_\pm^\nu \end{pmatrix} + \int \mathrm{d}p_z \frac{n_n(p_z)}{\varepsilon_n} \begin{pmatrix} 0 \\ k_\parallel^\mu \\ \pi^{\mu\nu} \end{pmatrix}, \tag{23}$$

where the dispersion is described in terms of the single combination of the integrals (21),

$$K_\pm(\omega, k_z) = \frac{\epsilon\varepsilon_n^0 I_{1\pm} + \varepsilon_\pm I_{2\pm}}{2g_{nn'}\varepsilon_n^0}, \tag{24}$$

with $p_\pm^\mu = (\varepsilon_\pm, 0, 0, p_{z\pm})$ and with

$$\pi^{\mu\nu} = \epsilon p_\parallel^\mu k_\parallel^\nu + \epsilon p_\parallel^\nu k_\parallel^\mu - \epsilon(pk)_\parallel g_\parallel^{\mu\nu} + \left(k_\parallel^\mu k_\parallel^\nu + k_D^\mu k_D^\nu\right) f_{nn'} \tag{25}$$

in the nondispersive term. The result (23) with (24) is consistent with a known property of dispersion integrals [22]: the numerator can be evaluated at resonance and taken outside the integral. However, a detailed evaluation is needed to determine the nondispersive term.

The response 4-tensor (17) reduces to a form

$$\Pi^{\mu\nu}(k) = \Pi_{\mathrm{ND}}^{\mu\nu}(k) - \frac{e^3 B}{8\pi^2\left(\omega^2 - k_z^2\right)} \sum_{\epsilon,n,n',\pm} \pm\left\{[N_{n'n}(p_\pm, k)]^{\mu\nu} \bar{K}_\pm(\omega, k_z)\right.$$
$$\left. + \epsilon[G_{n'n}(p_\pm, k)]^{\mu\nu} K_\pm^d(\omega, k_z)\right\}, \tag{26}$$

with $\Pi_{ND}^{\mu\nu}(k)$ being a nondispersive part, arising from the final integrals in (23). The dispersion functions $\bar{K}_\pm(\omega, k_z)$, $K_\pm^d(\omega, k_z)$ correspond to (24) with $n_n(p_z) \to \bar{n}_n(p_z), n_n^d(p_z)$, respectively, in (21); these functions depend implicitly on ϵ.

4.2. Nondispersive part

The nondispersive part involves the response tensor (25) with contributions from electrons and positrons in each Landau level. The integrals may be expressed in terms of the total proper number density, $\bar{n}_{\rm pr}$:

$$\bar{n}_{\rm pr} = \sum_{\epsilon=\pm}\sum_{n=0} g_n n^{\epsilon}_{n\rm pr}, \tag{27}$$

with the degeneracy factor $g_0 = 1$ and $g_n = 2$ for $n \geqslant 1$, and with

$$n^{\epsilon}_{n\rm pr} = \frac{eBm}{2\pi}\int\frac{{\rm d}p_z}{2\pi}\frac{n^{\epsilon}_n(p_z)}{\varepsilon_n}. \tag{28}$$

Using the sum rules, derived by Sokolov and Ternov [23],

$$\sum_{n'=0}^{\infty} J^{n'}_{n-n'}(x)J^{n'}_{n''-n'}(x) = \delta_{nn''}, \tag{29}$$

$$\sum_{n'=0}^{\infty}(n'-n)\big[J^{n'}_{n-n'}(x)\big]^2 = x, \tag{30}$$

one finds

$$\Pi^{\mu\nu}_{ND} = -\frac{e^2\bar{n}_{\rm pr}}{m}\left[g^{\mu\nu}_{\perp} - \frac{\left(k^{\mu}_{\parallel}k^{\nu}_{\parallel} + k^{\mu}_{D}k^{\nu}_{D}\right)k^2_{\perp}}{[(k^2)_{\parallel}]^2} - \frac{k^{\mu}_{\parallel}k^{\nu}_{\perp} + k^{\mu}_{\perp}k^{\nu}_{\parallel}}{(k^2)_{\parallel}}\right], \tag{31}$$

with $(k^2)_{\parallel} = \omega^2 - k_z^2$.

4.3. Dispersive part

The dispersive part of (26) is made up of two contributions, both summed over $\pm$: a nongyrotropic contribution $[N_{n'n}(p_{\pm},k)]^{\mu\nu}$ times $\bar{K}_{\pm}(\omega,k_z)$ and a gyrotropic contribution $[G_{n'n}(p_{\pm},k)]^{\mu\nu}$ times $K^d_{\pm}(\omega,k_z)$. The nongyrotropic contribution is

$$\begin{aligned}
[N_{n'n}(p_{\pm},k)]^{\mu\nu} = {}& \big\{g^{\mu\nu}_{\parallel}\big[\big(J^{n-1}_{n'-n}\big)^2 + \big(J^{n}_{n'-n}\big)^2\big] + g^{\mu\nu}_{\perp}\big[\big(J^{n-1}_{n'-n+1}\big)^2 \\
& + \big(J^{n}_{n'-n-1}\big)^2\big]\big\}\big[(p_{\pm}k)_{\parallel} - 2neB\big] \\
& + [2p^{\mu}_{\pm}p^{\nu}_{\pm} - (p^{\mu}_{\pm}k^{\nu}_{\parallel} + p^{\nu}_{\pm}k^{\mu}_{\parallel})]\big[\big(J^{n-1}_{n'-n}\big)^2 + \big(J^{n}_{n'-n}\big)^2\big] \\
& + 2p_{n'}p_n\big\{g^{\mu\nu}_{\parallel}J^{n-1}_{n'-n}J^{n}_{n'-n} + \big[e^{\mu}_1e^{\nu}_1 - e^{\mu}_2e^{\nu}_2\big]J^{n-1}_{n'-n+1}J^{n}_{n'-n-1}\big\} \\
& + p_n\big(k^{\mu}_{\parallel}e^{\nu}_1 + k^{\nu}_{\parallel}e^{\mu}_1\big)\big[J^{n-1}_{n'-n}J^{n}_{n'-n-1} + J^{n}_{n'-n}J^{n-1}_{n'-n+1}\big] \\
& - p_n\big[p^{\mu}_{\pm}e^{\nu}_1 + p^{\nu}_{\pm}e^{\mu}_1\big]\big[J^{n-1}_{n'-n}J^{n}_{n'-n-1} + J^{n}_{n'-n}J^{n-1}_{n'-n+1}\big] \\
& - p_{n'}\big[p^{\mu}_{\pm}e^{\nu}_1 + p^{\nu}_{\pm}e^{\mu}_1\big]\big[J^{n-1}_{n'-n}J^{n-1}_{n'-n+1} + J^{n}_{n'-n}J^{n}_{n'-n-1}\big],
\end{aligned} \tag{32}$$

and the gyrotropic contribution is

$$\begin{aligned}
[G_{n'n}(p_{\pm},k)]^{\mu\nu} = {}& {\rm i}\big\{f^{\mu\nu}[2neB - (p_{\pm}k)_{\parallel}]\big[\big(J^{n-1}_{n'-n+1}\big)^2 - \big(J^{n}_{n'-n-1}\big)^2\big] \\
& + p_n\big(k^{\mu}_{\parallel}e^{\nu}_2 - k^{\nu}_{\parallel}e^{\mu}_2\big)\big[J^{n-1}_{n'-n}J^{n}_{n'-n-1} - J^{n}_{n'-n}J^{n-1}_{n'-n+1}\big] \\
& - p_n\big[p^{\mu}_{\pm}e^{\nu}_2 - p^{\nu}_{\pm}e^{\mu}_2\big]\big[J^{n-1}_{n'-n}J^{n}_{n'-n-1} - J^{n}_{n'-n}J^{n-1}_{n'-n+1}\big] \\
& + p_{n'}\big[p^{\mu}_{\pm}e^{\nu}_2 - p^{\nu}_{\pm}e^{\mu}_2\big]\big[J^{n-1}_{n'-n}J^{n-1}_{n'-n+1} - J^{n}_{n'-n}J^{n}_{n'-n-1}\big]\big\}.
\end{aligned} \tag{33}$$

4.4. Plasma dispersion function

An alternative way of writing the plasma dispersion function (24) involves changing the variable of integration from p_z to t, defined by writing

$$\frac{p_z}{\varepsilon_n^0} = \frac{2t}{1-t^2}, \qquad \frac{\varepsilon_n}{\varepsilon_n^0} = \frac{1+t^2}{1-t^2}, \qquad \frac{p_{z\pm}}{\varepsilon_n^0} = \frac{2t_\pm}{1-t_\pm^2}, \qquad \frac{\varepsilon_\pm}{\varepsilon_n^0} = \frac{1+t_\pm^2}{1-t_\pm^2}. \tag{34}$$

The resonances in the form $\epsilon p_z = p_{z\pm}$ are reproduced by $\epsilon t = t_\pm, -1/t_\pm$ with

$$t_\pm = \frac{\varepsilon_\pm - \varepsilon_n^0}{p_{z\pm}}, \qquad \frac{1}{t_\pm} = \frac{\varepsilon_\pm + \varepsilon_n^0}{p_{z\pm}}. \tag{35}$$

One finds that the combination of functions in (24) can be expressed in the form

$$K_\pm(\omega, k_z) = \frac{1}{g_{nn'}} \int_{-1}^{1} \frac{\mathrm{d}t t}{1-t^2} n_n(t) + \frac{1}{g_{nn'}} \left[\frac{1}{2}(1+\epsilon) J(t_\pm) + \frac{1}{2}(1-\epsilon) J(1/t_\pm) \right], \tag{36}$$

where only a single plasma dispersion function,

$$J(t_0) = \int_{-1}^{1} \mathrm{d}t \frac{n_n(t)}{t - t_0}, \tag{37}$$

is required for either $n_n(t) = \bar{n}_n(t)$ or $n_n^d(t)$, where $n_n(t)$ denotes $n_n(p_z)$ with p_z expressed in terms of t through (34). The terms in (36) proportional to $\frac{1}{2}(1 \pm \epsilon)$ describe dispersion associated with gyromagnetic absorption and pair creation, respectively.

5. Completely degenerate limit

For a completely degenerate distribution the dispersion function (37) becomes a logarithmic function. In this section, we evaluate the response tensor in this case and show that it reproduces the known particular case of parallel propagation.

5.1. Completely degenerate limit

In the completely degenerate limit, the occupation number at each Landau level is unity up to the Fermi energy and zero above it. This corresponds to all the states being filled for $|p_z| < p_{nF}$, with

$$p_{nF} = \left(\varepsilon_F^2 - m^2 - 2neB\right)^{1/2}. \tag{38}$$

Only Landau levels $n < n_F$ are occupied, with $n = n_F$ being the maximum n for which p_{nF}, defined by (38), is real. The occupation number for electrons at the nth level is $n_n^+(t) = 1$, for $|t| < t_F = p_{nF}/(\varepsilon_F + \varepsilon_n^0)$, and $n_n^+(t) = 0$, for $|t| > t_F$. The occupation number for positrons is zero, so that we have $\bar{n}_n(t) = n_n^d(t) = n_n^+(t)$. With the upper and lower limits of ± 1 on the integrals over t replaced by t_F, the plasma dispersion function (37) becomes

$$J(t_0) = \ln\left|\frac{t_F - t_0}{t_F + t_0}\right| = \ln\left|\frac{p_{nF} - \left(\varepsilon_F + \varepsilon_n^0\right)t_0}{p_{nF} + \left(\varepsilon_F + \varepsilon_n^0\right)t_0}\right|, \tag{39}$$

with t_0 identified as either $t_\pm$ or $1/t_\pm$. The resulting logarithmic factors may be rewritten using the identities

$$\left|\frac{p_{z\pm}\varepsilon_F - p_{nF}\varepsilon_\pm}{p_{z\pm}\varepsilon_F + p_{nF}\varepsilon_\pm}\right| = \left|\frac{(t_F - t_\pm)(t_F - 1/t_\pm)}{(t_F + t_\pm)(t_F + 1/t_\pm)}\right|, \qquad \left|\frac{p_{z\pm} - p_{nF}}{p_{z\pm} + p_{nF}}\right| = \left|\frac{(t_F - t_\pm)(t_F + 1/t_\pm)}{(t_F + t_\pm)(t_F - 1/t_\pm)}\right|,$$

so that one has

$$J(t_\pm) + J(1/t_\pm) = \ln\left|\frac{p_{z\pm}\varepsilon_F - p_{nF}\varepsilon_\pm}{p_{z\pm}\varepsilon_F + p_{nF}\varepsilon_\pm}\right|, \qquad J(t_\pm) - J(1/t_\pm) = \ln\left|\frac{p_{z\pm} - p_{nF}}{p_{z\pm} + p_{nF}}\right|. \tag{40}$$

With the first term in (36) zero, after summing over ϵ, (26) gives

$$\Pi^{\mu\nu}(k) = \Pi^{\mu\nu}_{\rm ND}(k) - \frac{e^3 B}{8\pi^2(\omega^2 - k_z^2)} \sum_{n,n',\pm} \frac{\pm 1}{g_{n'n}} \times \left\{ [N_{n'n}(p_\pm, k)]^{\mu\nu} \ln\left(\frac{p_{z\pm}\varepsilon_F - p_{nF}\varepsilon_\pm}{p_{z\pm}\varepsilon_F + p_{nF}\varepsilon_\pm}\right) + [G_{n'n}(p_\pm, k)]^{\mu\nu} \ln\left|\frac{p_{z\pm} - p_{nF}}{p_{z\pm} + p_{nF}}\right| \right\}, \tag{41}$$

with the nondispersion term given explicitly by

$$\Pi^{\mu\nu}_{\rm ND}(k) = -\frac{{\rm e}^3 B}{4\pi^2} \sum_{n=0} g_n \ln\left|\frac{\varepsilon_F + p_{nF}}{\varepsilon_F - p_{nF}}\right| \left[g_\perp^{\mu\nu} - \frac{\left(k_\parallel^\mu k_\parallel^\nu + k_D^\mu k_D^\nu\right)k_\perp^2}{[(k^2)_\parallel]^2} - \frac{k_\parallel^\mu k_\perp^\nu + k_\perp^\mu k_\parallel^\nu}{(k^2)_\parallel} \right], \tag{42}$$

where the logarithmic factor is proportional to the proper number density in the nth Landau level. The logarithmic functions in (41) are the same as in the case of parallel propagation [18], confirming that no additional dispersive effects are included by the generalization to oblique propagation.

Explicit evaluation of the non-gyrotropic and gyrotropic parts in (41) is facilitated by choosing $\psi = 0$, so that $[G_{n'n}(p_\pm, k)]^{\mu\nu}$ is non-zero for $\mu \neq 2, \nu = 2$; $\mu = 2, \nu \neq 2$ and $[N_{n'n}(p_\pm, k)]^{\mu\nu}$ is non-zero for the remaining $\mu\nu$ values. Explicit forms are ($\mu = 0, 3$ only in the following)

$$\begin{aligned}
[N_{n'n}(p_\pm, k)]^{00} &= [2\varepsilon_\pm^2 - \varepsilon_\pm\omega - p_{z\pm}k_z - 2neB]\left[\left(J^{n-1}_{n'-n}\right)^2 + \left(J^n_{n'-n}\right)^2\right] + 2p_{n'}p_n J^{n-1}_{n'-n}J^n_{n'-n},\\
[N_{n'n}(p_\pm, k)]^{11} &= -[(p_\pm k)_\parallel - 2neB]\left[\left(J^{n-1}_{n'-n+1}\right)^2 + \left(J^n_{n'-n-1}\right)^2\right] + 2p_{n'}p_n J^{n-1}_{n'-n+1}J^n_{n'-n-1},\\
[N_{n'n}(p_\pm, k)]^{22} &= -[(p_\pm k)_\parallel - 2neB]\left[\left(J^{n-1}_{n'-n+1}\right)^2 + \left(J^n_{n'-n-1}\right)^2\right] - 2p_{n'}p_n J^{n-1}_{n'-n+1}J^n_{n'-n-1},\\
[N_{n'n}(p_\pm, k)]^{33} &= \left[2p_{z\pm}^2 - \varepsilon_\pm\omega - p_{z\pm}k_z + 2neB\right]\left[\left(J^{n-1}_{n'-n}\right)^2 + \left(J^n_{n'-n}\right)^2\right] - 2p_{n'}p_n J^{n-1}_{n'-n}J^n_{n'-n},\\
[N_{n'n}(p_\pm, k)]^{03} &= [2\varepsilon_\pm p_{z\pm} - (\varepsilon_\pm k_z + p_{z\pm}\omega)]\left[\left(J^{n-1}_{n'-n}\right)^2 + \left(J^n_{n'-n}\right)^2\right],\\
[N_{n'n}(p_\pm, k)]^{\mu 1} &= p_n\left(k_\parallel^\mu - p_\pm^\mu\right)\left[J^{n-1}_{n'-n}J^n_{n'-n-1} + J^n_{n'-n}J^{n-1}_{n'-n+1}\right]\\
&\quad - p_{n'}p_\pm^\mu\left[J^{n-1}_{n'-n}J^{n-1}_{n'-n+1} + J^n_{n'-n}J^n_{n'-n-1}\right],\\
[G_{n'n}(p_\pm, k)]^{\mu 2} &= {\rm i}\left\{p_n\left(k_\parallel^\mu - p_\pm^\mu\right)\left[J^{n-1}_{n'-n}J^n_{n'-n-1} - J^n_{n'-n}J^{n-1}_{n'-n+1}\right]\right.\\
&\quad \left. + p_{n'}p_\pm^\mu\left[J^{n-1}_{n'-n}J^{n-1}_{n'-n+1} - J^n_{n'-n}J^n_{n'-n-1}\right]\right\},\\
[G_{n'n}(p_\pm, k)]^{12} &= -{\rm i}\left\{[2neB - (p_\pm k)_\parallel]\left[\left(J^{n-1}_{n'-n+1}\right)^2 - \left(J^n_{n'-n-1}\right)^2\right]\right\},
\end{aligned} \tag{43}$$

with the remaining components determined by the Hermitian condition.

6. Nondegenerate limit

The plasma dispersion function (37) may be written down for any distribution, including an arbitrary thermal distribution. In this section, after writing down a Fermi–Dirac distribution, we compare our results with known plasma dispersion function (37) for nondegenerate distribution functions.

6.1. Fermi–Dirac distribution

The most general form for a thermal electron distribution is a Fermi–Dirac (FD) distribution. This corresponds to the occupation number

$$n_n^\epsilon(p_z) = \frac{1}{{\rm e}^{(\varepsilon_n - \mu^\epsilon)/T} + 1}, \tag{44}$$

where T is the temperature (in energy units) and μ^ϵ are the chemical potentials for the electrons and positrons, respectively, with $\mu^+ + \mu^- = 0$. Although the plasma dispersion function can be written down for the FD-distribution (44), we are aware of no results in the literature with which we can compare the result. The completely degenerate limit, discussed above, corresponds to $T \to 0$ in the FD distribution (44), with the Fermi energy being equal to the chemical potential. Here we consider the nondegenerate limit.

6.2. Jüttner distribution

In the nondegenerate limit, the chemical potential $\mu^+ - m$ is large and negative. The unit term in the denominator in the FD distribution (44) is then negligible, and the resulting distribution reduces to a sum of one-dimensional Jüttner distributions,

$$n_n^\epsilon(p_z) = A_n^\epsilon \exp\left[-\rho\left(1 + \frac{p_z^2}{m^2} + 2n\frac{B}{B_c}\right)^{1/2}\right], \qquad \rho = \frac{m}{T}, \tag{45}$$

with $A_n^\epsilon = \exp(\rho\mu^\epsilon/m)$ related to the proper number density, n_{npr}^ϵ, in each Landau level by

$$\frac{g_n n_{npr}^\epsilon}{A_n^\epsilon} = g_n \frac{eBm}{2\pi^2} K_0\big(\rho\varepsilon_n^0/m\big), \tag{46}$$

with $g_0 = 1$, $g_n = 2$ for $n \geqslant 1$ and where K_ν are the modified Bessel functions [24].

For the Jüttner distribution (45), the plasma dispersion function (37) becomes

$$J_n^\epsilon(t_0) = A_n^\epsilon \int_{-1}^{1} \mathrm{d}t \frac{1}{t - t_0} \exp\left(-\frac{\varepsilon_n^0}{T}\frac{1 + t^2}{1 - t^2}\right). \tag{47}$$

This function may be evaluated in terms of a relativistic plasma dispersion function $T(v, \rho)$, used in the unmagnetized case [25]. In the magnetized case, it is convenient to write

$$T(v_0, \rho_n) = \int_{-1}^{1} \frac{\mathrm{d}v}{v - v_0} \exp(-\rho_n\gamma), \tag{48}$$

with $\gamma = (1 - v^2)^{-1/2}$, $\rho_n = \varepsilon_n^0\big/T$ and $v_0 = 2t_0\big/\big(1 + t_0^2\big)$. Using the properties of $T(v, \rho)$ in [25], one finds

$$J_n^\epsilon(t_0) = \frac{A_n^\epsilon}{2}\left[-\frac{\big(1 - v_0^2\big)^{1/2}}{v_0}\left(2K_1(\rho_n) + \frac{\big(1 - v_0^2\big)}{\rho_n} T'(v_0, \rho_n)\right) + T(v_0, \rho_n)\right], \tag{49}$$

with $T'(v_0, \rho_n) = \partial T(v_0, \rho_n)/\partial v_0$. The description of dispersion in a relativistic quantum magnetized nondegenerate thermal electron gas is characterized by the four dispersion functions given by (49) with $t_0 = t_\pm, 1/t_\pm$. An explicit form for the response tensor is closely analogous to (41): one replaces the logarithmic factors by the combinations implied by (40) and (49). There is no known explicit results with which to compare this general result.

6.3. Ultrarelativistic quantum limit

The response tensor in the ultrarelativistic limit for a nondegenerate thermal distribution was discussed by [26], who considered only the case where the particles are in their lowest Landau orbital, $n = 0$. Following [26], we assume that the thermal distribution function has a low-energy cutoff, (45), say at $\gamma = \gamma^* \gg 1$, so that all particles have $\gamma = (1 + t^2)/(1 - t^2) \gg 1$. The ultrarelativistic assumption corresponds to $1 - |t| \ll 1$. The allowed range of t is $t^* < |t| < 1$, with $t^* = 1 - 1/\gamma^*$ to first order in $1/\gamma^*$. In the following we allow n to have any value, with γ interpreted as $\varepsilon_n\big/\varepsilon_n^0$, so that ε_n^0 plays the role of an effective mass in the following analysis.

In this case, the plasma dispersion function (37) becomes

$$J_n^{\epsilon}(t_0) = A_n^{\epsilon} v_0 \int_{\gamma^*}^{\infty} \mathrm{d}\gamma \left[-\frac{1}{\gamma} + \frac{1}{\gamma - \gamma_0} \right] \exp\left(-\frac{\varepsilon_n^0}{T} \gamma \right), \tag{50}$$

with $v_0 = 2t_0 \big/ \left(1 + t_0^2\right)$, $\gamma_0 = \left(1 + t_0^2\right) \big/ \left(1 - t_0^2\right)$ not necessarily ultrarelativistic. The integral in (50) can be written in terms of the Dnestrovskii function [27, 28]

$$F_q(z) = \mathrm{e}^z \int_z^{\infty} \mathrm{d}y y^{-q} \mathrm{e}^{-y}, \tag{51}$$

with $q = 1$. This gives

$$J_n^{\epsilon}(t_0) = A_n^{\epsilon} v_0 \mathrm{e}^{-a_n \gamma^*} [F_1(a_n(\gamma^* - \gamma_0)) - F_1(a_n \gamma^*)], \tag{52}$$

with $a_n = \varepsilon_n^0 / T$. The dispersion function $F_1(a_n(\gamma^* - \gamma_0))$ reduces to the plasma dispersion function found by [26] for $n = 0$, who noted that the expansion

$$F_1(z) = -\mathrm{e}^z \left[\gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-z)^n}{n n!} \right], \tag{53}$$

with $\gamma = 0.5772\ldots$ Euler's constant, is analytic except at $z = 0$.

Further comparison with the work of [26] is complicated by different assumptions made in deriving the resonance conditions, here in the form $p_z = p_{z\pm}$. Our derivation of $p_{z\pm}$, and hence of $t_\pm$ in the dispersion function (52), is based on the quadratic equation obtained from the product of all four resonant denominators, leading to (8), with the solutions following from (9)–(11) and (34). The derivations of [26] involve starting from the product of only two of the resonant denominators, with the ultrarelativistic approximation used to find the solutions. The two approaches are equivalent for the leading term in the ultrarelativistic approximation only for both $\gamma \gg 1$ and $\gamma_0 \gg 1$, with γ_0 determined by one of $t_0 = t_\pm$, $1/t_\pm$ in (52). In this sense, (52) reproduces the plasma dispersion function of [26] when the resonance is in the ultrarelativistic range and provides a generalization to include contributions from resonances that are not in the ultrarelativistic range.

6.4. *Nonrelativistic quantum limit*

In the nonrelativistic limit, the thermal distribution function is a Maxwellian, which is given by setting $t = p_z / 2\varepsilon_n^0 \ll 1$ in (45). The plasma dispersion function (37) then reduces to

$$J_n^{\epsilon}(t_0) = \pi^{1/2} A_n^{\epsilon} \mathrm{e}^{-\varepsilon_n^0 / T} Z(y_n), \qquad y_n = \left(\frac{2\varepsilon_n^0}{T} \right)^{1/2} t_0, \tag{54}$$

where $Z(y)$ is the familiar Fried and Conte function [29], defined by

$$Z(y) = \pi^{-1/2} \int_{-\infty}^{\infty} \mathrm{d}x \frac{\mathrm{e}^{-x^2}}{x - y}. \tag{55}$$

The dielectric tensor for a nonrelativistic, magnetized quantum plasma is known [30]: it involves a sum over Landau levels, n, and an integral over p_z, with the electron distribution described by an arbitrary occupation number, $n_n(p_z)$. It was shown [30] that this result reproduces earlier known results for the completely degenerate (nonrelativistic) limit. Although it is straightforward to evaluate the integral (over p_z) for a 1D Maxwellian distribution in terms of the function (55), we are unaware of the result being written down explicitly.

Comparison of our relativistic quantum result with this nonrelativistic quantum result is relatively straightforward, except that the forms of the quantum recoil term are not the same. This is due to a subtle inconsistency. The expression obtained for the quantum recoil is different depending on when the nonrelativistic approximation is made: the nonrelativistic limit of the relativistically correct derivation of the recoil is different from that obtained by a strictly nonrelativistic derivation of the recoil. The nonrelativistic limit of the relativistically correct derivation of the recoil involves a factor (in ordinary units) $\omega^2/c^2 - k_z^2$, whereas a strictly nonrelativistic derivation gives $-k_z^2$. As a result, the nonrelativistic limit of expressions (9) with (11) does not reproduce the nonrelativistic result obtained by identifying $p_z = p_{z\pm}$ by setting $\omega \mp (\varepsilon_n - \varepsilon'_{n'}) = 0$ in the nonrelativistic limit, $\varepsilon_n = m + p_z^2/2m + n\Omega_0$, $\varepsilon'_{n'} = m + (p_z - k_z)^2/2m + n'\Omega_0$, with $\Omega_0 = eB/m$. In the strictly nonrelativistic approximation, the values of $t_\pm$ are to be interpreted as $p_{z\pm}/2m$, with $p_{z\pm}$ determined in this way. As a consequence, the nonrelativistic limit of our result does not reproduce the recoil term in the resonant denominator in the known nonrelativistic quantum results [30].

7. Discussion

Our main purpose in this paper is to derive a general expression for the response 4-tensor for a spin-independent, relativistic quantum magnetized electron gas. The quantum effects included are quantization of the electron motion perpendicular to the field lines to give the Landau levels, $n = 0, 1, \ldots$, the quantum recoil, degeneracy and dispersion due to one-photon pair creation. The first three of these have nonrelativistic counterparts; dispersion due to pair creation is an intrinsically relativistic quantum effect. A direct method of derivation of this response tensor is to start from a known expression for the response tensor (1), that involves spin-dependent vertex functions, and perform the sums over spin states explicitly. An alternative derivation is analogous to the derivation of the spin-independent form in the unmagnetized case, where the sum over spins is replaced by traces over Dirac matrices. We develop such a method, which we refer to as the Ritus method (it has also been referred to as the Parle method [31]), and use it to complement the direct method.

Another technique that facilities simplifying the general form is to solve the relativistic quantum resonance condition in (1) for the resonant values, $\epsilon p_z = p_{z\pm}$ [13], and to construct the associated particle 4-momentum, $p_\pm^\mu$. Rationalization of the resonant denominator then allows one to sum over the electron and positron contributions to the virtual intermediate state. The resulting alternative general form (17) for the response tensor separates naturally into nongyrotropic and gyrotropic parts, which are even and odd, respectively, under interchange of electrons and positrons. Another simplification is that the dispersion integrals, which involves a numerator that is a tensor that depends on p^μ and a resonant denominator, can be rewritten such that the numerator is evaluated at $p_\pm^\mu$ and can be taken outside the p_z-integral. This way of evaluating dispersion integrals is suggested by a general property of dispersion integrals, due to Cutkovsky [22], which implies that a dispersion integral whose integrand is of the form N/D can be expressed in terms of the values of the numerator, N, evaluated at the resonant values implied by the vanishing of the denominator, $D = 0$. The resulting form (26) involves a nondispersive part (31) and a dispersive part that depends on the plasma dispersion function evaluated at four arguments, $t_0 \to t_\pm, 1/t_\pm$, with $p_{z\pm}\big/\varepsilon_n^0 = 2t_\pm/(1 - t_\pm^2)$. For an arbitrary electron distribution, the plasma dispersion function is defined for the sum and difference, $n_n(t) \to \bar{n}_n(t), n_n^d(t)$, of the occupation numbers for the electrons and positrons at each Landau level. We write down the plasma dispersion function for an arbitrary thermal (Fermi–Dirac) distribution, and evaluate it explicitly in the completely degenerate and nondegenerate limits.

The reduction of the relativistic quantum form to the known (covariant) nonquantum limit [32] is relatively straightforward when one starts from the form (1) with (2). One has (in SI units) $\hbar \to 0, n \to \infty$, with $\hbar n \to p_\perp^2/2eB$; the sum over n is replaced by an integral over $p_\perp^2/2eB$; the function $J_\nu^n(x)$ with $x = \hbar k_\perp^2/2eB$ reduces to the Bessel function $J_\nu(k_\perp p_\perp/eB)$ to lowest order in an expansion in $\hbar$ (and in $1/n$); with this approximation to the J-functions, the vertex functions in (2) reduce to their nonquantum counterparts; in addition, one neglects the vacuum contribution and retains only the contribution from electrons. The dispersion functions arise from p_z-integral over the resonant denominators. Two resonant denominators are approximated by $\omega \mp (\varepsilon_n - \varepsilon'_{n'}) \to \omega \mp [(n - n')eB + k_z p_z]/\varepsilon$; these led to dispersion integrals associated with gyromagnetic absorption in the limit where the quantum recoil is neglected. The two denominators associated with pair creation, $\omega \mp (\varepsilon_n + \varepsilon'_{n'})$, are approximated by $\mp 2\left(m^2 + p_\perp^2 + p_z^2\right)^{1/2}$; although there is no contribution from dispersion due to pair creation in the nonquantum limit, one needs to retain these terms ($\epsilon = 1 = -\epsilon'$) in (1) to reproduce the nondispersive part correctly.

A corollary of the result that only one plasma dispersion function is needed in the relativistic quantum case, is that only one dispersion function is needed in the nonquantum case. It is straightforward to show that this is the case in the nonrelativistic, nonquantum result. The long-known expression for the response tensor in this limit [30] involves p_z-integrals over the occupation number, $n_n(p_z)$, with resonant denominators $\omega \mp [(n - n')eB + k_z p_z]/m$. For an arbitrary $n_n(p_z)$ this integral defines a (nonrelativistic-nonquantum) dispersion function; for example, these can be evaluated in terms of the familiar plasma dispersion function (55) for a Maxwellian distribution. The corollary implies that only one dispersion function is required in the relativistic case, and this is not so obvious because of the square root in the denominator associated with $\varepsilon = (m^2 + p_\perp^2 + p_z^2)^{1/2}$. However, no rationalization is required if one changes the variable of integration from p_z to t, defined by $p_z = (m^2 + p_\perp^2)^{1/2} 2t/(1 - t^2)$ in analogy with (34). In this case, the resonant denominators become quadratic functions of t, and nonquantum counterparts of the roots $t = t_\pm$ can be identified. In this way, the dispersion integrals can be written in terms of the plasma dispersion function (37) evaluated at the relevant nonquantum counterparts of $t_\pm$, which depend on $p_\perp$. The nonquantum limit includes an integral over $p_\perp$, and because the plasma dispersion function now depends on this variable, the nonquantum limit in this approach is considerably more cumbersome than the quantum case, where the parameter n is discrete. For a one-dimensional distribution with $p_\perp = 0$ this difficulty does not arise, and it is straightforward to show that the known nonquantum result [32] is reproduced. Whether or not this alternative way of evaluating the dispersion integrals is useful more generally in the nonquantum case warrants further investigation. In the context of the nonquantum limit of our relativistic quantum result, we note that the relevant expressions have yet to be derived by nonquantum methods.

The response 4-tensor for a completely degenerate electron gas reproduces the known results for parallel propagation [14, 16, 18] and provides the generalization to arbitrary angles of propagation. The generalization to oblique angles involves a major increase in algebraic complexity, notably through the dependence on the J-functions, but introduces no new dispersive features, which depend on ω, k_z, but not on $k_\perp$. In particular, the boundaries (in the ω–k_z-plane) for the different resonant contributions are unchanged from the case of parallel propagation [18]. Our explicit expression for the response tensor for oblique propagation for a completely degenerate distribution (section 5) also applies to an arbitrary distribution, provided that one replaces the logarithmic plasma dispersion function by the appropriate plasma dispersion function, determined by (37). We write down an explicit form, (49), for this dispersion function for a relativistic thermal (1D Jüttner) distribution in terms of a known relativistic plasma dispersion function. We also compare this plasma dispersion function with known expressions for the ultrarelativistic and nonrelativistic limits.

8. Conclusions

The main result presented in this paper is a general expression for the response tensor for a magnetized, relativistic quantum electron gas summed over spin states and over virtual electron and positron states. Although general forms for the response tensor have long been known, these are cumbersome to use because of the explicit dependence on the choice of spin operator, with the sum over virtual states leading to contributions involving all possible spin transitions and from electron and positron states. We introduce several different techniques to perform these sums explicitly. Most notable is the development of a method due to Ritus [21] which allows the sums to be performed in a covariant manner without introducing a spin operator explicitly. We show that the general result can be expressed in terms of resonant values $(p_{z\pm}, \varepsilon_\pm, t_\pm)$ and a single relativistic plasma dispersion function. We make a comparison of this general result with some known special cases. The comparison is straightforward only for parallel propagation for a completely degenerate distribution. A new specific result in this paper is the generalization of the response tensor to arbitrary angles of propagation for a completely degenerate distribution. The plasma dispersion function, which is logarithmic in this case, is independent of the angle of propagation. This explicit result generalizes further to an arbitrary distribution simply by replacing the logarithmic function by the general form for the plasma dispersion function. We write down the form of the plasma dispersion function for a nondegenerate thermal (one-dimensional Jüttner) distribution and show how it reproduces a known result in the ultrarelativistic limit [26]. We also comment on the known nonrelativistic quantum form for the response 3-tensor [30].

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Appendix A. Vertex function

An explicit form for the vertex function that appears in (1) requires a specific choice of the spin operator. For the choice of the magnetic momentum operator [15, 23], one has

$$
\begin{aligned}
&\left[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})\right]^\mu = (\mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi})^{n-n'}\left(A^{\epsilon'\epsilon(+)}_{q'q}J^{(0)}_{q'q}(\mathbf{k}),\, -A^{\epsilon'\epsilon(-)}_{q'q}J^{(+)}_{q'q}(\mathbf{k}),\, -i\,A^{\epsilon'\epsilon(-)}_{q'q}J^{(-)}_{q'q}(\mathbf{k}),\, B^{\epsilon'\epsilon}_{q'q}J^{(0)}_{q'q}(\mathbf{k})\right),\\
&A^{\epsilon'\epsilon(\pm)}_{q'q} = a'_{\epsilon's'}a_{\epsilon s} \pm a'_{-\epsilon's'}a_{-\epsilon s}, \qquad B^{\epsilon'\epsilon}_{q'q} = a'_{\epsilon's'}a_{-\epsilon s} + a'_{-\epsilon's'}a_{\epsilon s},\\
&J^{(0)}_{q'q}(\mathbf{k}) = b'_{s'}b_s J^{n-1}_{n'-n}(x) + s'sb'_{-s'}b_{-s}J^{n}_{n'-n}(x),\\
&J^{(\pm)}_{q'q}(\mathbf{k}) = s'b'_{-s'}b_s\mathrm{e}^{-\mathrm{i}\psi}J^{n-1}_{n'-n+1}(x) \pm sb'_{s'}b_{-s}\mathrm{e}^{\mathrm{i}\psi}J^{n}_{n'-n-1}(x),\\
&a_\pm = P_\pm\left(\frac{\varepsilon_n \pm \varepsilon^0_n}{2\varepsilon_n}\right)^{1/2}, \qquad b_s = \left(\frac{\varepsilon^0_n + sm}{2\varepsilon^0_n}\right)^{1/2},
\end{aligned}
\tag{A.1}
$$

with $x = k_\perp^2/2eB$, and $P_\pm = \frac{1}{2}(1+P) \pm \frac{1}{2}(1-P)$, $P = p_z/|p_z|$. The J-functions are defined by (5) with $L^\nu_n(x)$ the generalized Laguerre polynomial,

$$
L^\nu_n(x) = \frac{\mathrm{e}^x x^{-\nu}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+\nu}) = \sum_{m=0}^{n}(-)^m\binom{n+\nu}{n-m}\frac{x^m}{m!}. \tag{A.2}
$$

Appendix B. Ritus method: evaluation of traces

In the Ritus method, the Dirac wavefunction is factorized in the form

$$\Psi_q^\epsilon(x) = \mathrm{e}^{-\mathrm{i}\epsilon(\varepsilon_n t - p_z z)}\mathcal{V}_g^\epsilon(\mathbf{x}, n, p_z)\varphi_s^\epsilon(n, p_z), \tag{B.1}$$

where $\mathcal{V}_g^\epsilon(\mathbf{x}, n, p_z)$ is a diagonal matrix, with g being a gauge-dependent quantum number, whose explicit form is not needed here. The reduced wavefunction may be written as

$$\varphi_s^\epsilon(n, p_z) = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}. \tag{B.2}$$

The column matrix in (B.2) may be interpreted as a reduced Dirac wavefunction, which satisfies a reduced Dirac equation, which (in the standard representation) has the explicit form

$$\begin{pmatrix} \epsilon\varepsilon_n - m & 0 & -\epsilon p_z & \mathrm{i}p_n \\ 0 & \epsilon\varepsilon_n - m & -\mathrm{i}p_n & \epsilon p_z \\ -\epsilon p_z & \mathrm{i}p_n & \epsilon\varepsilon_n + m & 0 \\ -\mathrm{i}p_n & \epsilon p_z & 0 & \epsilon\varepsilon_n + m \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 0. \tag{B.3}$$

The factorization (B.1) allows the vertex function to be written as

$$\left[\Gamma_{q'q}^{\epsilon'\epsilon}(\mathbf{k})\right]^\mu = V\bar{\varphi}_{s'}^{\epsilon'}(n', p_z')\mathcal{J}_{n'n}^\mu(\mathbf{k}_\perp)\varphi_s^\epsilon(n, p_z), \tag{B.4}$$

which defines the vertex matrix $\mathcal{J}_{n'n}^\mu(\mathbf{k}_\perp)$. It is useful to project onto the $\parallel$ and $\perp$-subspaces, by writing $\gamma_\parallel^\mu = g_\parallel^{\mu\nu}\gamma_\nu$, $\gamma_\perp^\mu = g_\perp^{\mu\nu}\gamma_\nu$, so that these correspond to $\gamma_\parallel^\mu = (\gamma^0, 0, 0, \gamma^3)$, $\gamma_\perp^\mu = (0, \gamma^1, \gamma^2, 0)$. One has

$$\mathcal{J}_{n'n}^\mu(\mathbf{k}_\perp) = \gamma_\parallel^\mu\mathcal{J}_{n'n}^\parallel(\mathbf{k}_\perp) + \gamma_\perp^\mu\mathcal{J}_{n'n}^\perp(\mathbf{k}_\perp), \tag{B.5}$$

where $\mathcal{J}_{n'n}^\parallel(\mathbf{k}_\perp)$, $\mathcal{J}_{n'n}^\perp(\mathbf{k}_\perp)$ are diagonal 4×4 matrices:

$$\mathcal{J}_{n'n}^\parallel(\mathbf{k}_\perp) = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n}\begin{pmatrix} J_{n'-n}^{n-1} & 0 & 0 & 0 \\ 0 & J_{n'-n}^n & 0 & 0 \\ 0 & 0 & J_{n'-n}^{n-1} & 0 \\ 0 & 0 & 0 & J_{n'-n}^n \end{pmatrix}, \tag{B.6}$$

$$\mathcal{J}_{n'n}^\perp(\mathbf{k}_\perp) = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n} \times \begin{pmatrix} -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J_{n'-n+1}^{n-1} & 0 & 0 & 0 \\ 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J_{n'-n-1}^n & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J_{n'-n+1}^{n-1} & 0 \\ 0 & 0 & 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J_{n'-n-1}^n \end{pmatrix}, \tag{B.7}$$

where the argument of the J-functions is $k_\perp^2/2eB$ and $\mathbf{k}_\perp = k_\perp(\cos\psi, \sin\psi, 0)$.

The matrices $\gamma_\parallel^\mu$, $\mathcal{J}_{n'n}^\parallel(\mathbf{k}_\perp)$ commute, but the matrices $\gamma_\perp^\mu$, $\mathcal{J}_{n'n}^\perp(\mathbf{k}_\perp)$ do not. If one writes the matrix products in (B.5) in the opposite order, one needs to replace $\mathcal{J}_{n'n}^\perp(\mathbf{k}_\perp)$ by

$$\tilde{\mathcal{J}}_{n'n}^\perp(\mathbf{k}_\perp) = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n} \times \begin{pmatrix} \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J_{n'-n-1}^n & 0 & 0 & 0 \\ 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J_{n'-n+1}^{n-1} & 0 & 0 \\ 0 & 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J_{n'-n-1}^n & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J_{n'-n+1}^{n-1} \end{pmatrix}. \tag{B.8}$$

The matrix (B.8) also appears in the symmetry relations

$$\mathcal{J}^{\parallel}_{nn'}(-\mathbf{k}_\perp) = [\mathcal{J}^{\parallel}_{n'n}(\mathbf{k}_\perp)]^*, \qquad \mathcal{J}^{\perp}_{nn'}(-\mathbf{k}_\perp) = [\tilde{\mathcal{J}}^{\perp}_{n'n}(\mathbf{k}_\perp)]^*. \tag{B.9}$$

Using relations (B.9) in (B.5), one finds

$$\mathcal{J}^{\mu}_{nn'}(-\mathbf{k}_\perp) = \gamma^\mu_\parallel[\mathcal{J}^{\parallel}_{n'n}(\mathbf{k}_\perp)]^* + \gamma^\mu_\perp[\tilde{\mathcal{J}}^{\perp}_{n'n}(\mathbf{k}_\perp)]^*. \tag{B.10}$$

An alternative way of writing $\mathcal{J}^{\mu}_{n'n}(\mathbf{k}_\perp)$ is in terms of the projection matrices $\mathcal{P}_\pm = \frac{1}{2}[1 \pm \Sigma_z]$, with Σ_z diagonal $(1, -1, 1, -1)$. One has

$$\begin{aligned}\mathcal{J}^{\mu}_{n'n}(\mathbf{k}_\perp) = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n}\big\{&\gamma^\mu_\parallel\big[J^{n-1}_{n'-n}(x)\mathcal{P}_+ + J^{n}_{n'-n}(x)\mathcal{P}_-\big]\\ &+\gamma^\mu_\perp\big[-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J^{n-1}_{n'-n+1}(x)\mathcal{P}_+ + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J^{n}_{n'-n-1}(x)\mathcal{P}_-\big]\big\}.\end{aligned} \tag{B.11}$$

These matrices satisfy the identities

$$\mathcal{P}_\pm\gamma^\mu_\parallel = \gamma^\mu_\parallel\mathcal{P}_\pm, \qquad \mathcal{P}_\pm\gamma^\mu_\perp = \gamma^\mu_\perp\mathcal{P}_\mp. \tag{B.12}$$

Using these relations, an alternative form of (B.11), with the matrix products written in the opposite order, is

$$\begin{aligned}\mathcal{J}^{\mu}_{n'n}(\mathbf{k}_\perp) = (-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi})^{n'-n}\big\{&\big[J^{n-1}_{n'-n}(x)\mathcal{P}_+ + J^{n}_{n'-n}(x)\mathcal{P}_-\big]\gamma^\mu_\parallel\\ &+\big[-\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\psi}J^{n-1}_{n'-n+1}(x)\mathcal{P}_- + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}J^{n}_{n'-n-1}(x)\mathcal{P}_+\big]\gamma^\mu_\perp\big\}.\end{aligned} \tag{B.13}$$

The sum over spins is replaced, in the Ritus method, by a trace over a product of Dirac matrices. The basic sum used is

$$\sum_{s=\pm}\varphi^\epsilon_s(n, p_z)\bar{\varphi}^\epsilon_s(n, p_z) = \frac{\not{P}^\epsilon_n + m}{2\epsilon\varepsilon_n V}, \qquad \big[P^\epsilon_n\big]^\mu = (\epsilon\varepsilon_n, 0, p_n, \epsilon p_z). \tag{B.14}$$

The evaluation of traces is similar to the unmagnetized case, with the added complication that projection matrices are included. Evaluation of the trace of a product of two γ-matrices becomes

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\mathcal{P}_\pm] = 2[g^{\mu\nu} \pm \mathrm{i}\,f^{\mu\nu}], \tag{B.15}$$

and the trace of a product of four γ-matrices becomes

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\mathcal{P}_\pm] = {}&2[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\rho\nu}]\\ &\pm 2\mathrm{i}[g^{\mu\nu}f^{\rho\sigma} + f^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}f^{\nu\sigma} - f^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}f^{\nu\rho} + f^{\mu\sigma}g^{\nu\rho}].\end{aligned} \tag{B.16}$$

The projection operator in (B.15) or (B.16) may be moved to any other location using the commutation relations $\mathcal{P}_\pm M_\parallel = M_\parallel\mathcal{P}_\pm$, with $M_\parallel = \gamma^0$ or γ^3, or $\mathcal{P}_\pm M_\perp = M_\perp\mathcal{P}_\mp$, with $M_\perp = \gamma^1$ or γ^2.

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